Geometry and Invariants

of the Affine Group

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CONTENTS

1. INTRODUCTION

Let k be an algebraically closed field of characteristic zero and V a finite dimensional k-vector space. Denote by $\mathrm{Aff}(V)$ the group of *affine transformations* of V, i.e. the maps $\varphi : V \to V$ of the form $\varphi(a) = ga + v$ where $g \in \mathrm{GL}(V)$ and $v \in V$. It is isomorphic to $V^+ \rtimes \mathrm{GL}(V)$. Its Lie algebra $\mathfrak{aff}(V)$ is isomorphic to $V \rtimes \mathrm{End}(V)$. Viewed as variety, V is equal to the affine n-space $\mathbb{A}_k^n$ and $\mathrm{Aff}(V)$ acts naturally on $\mathbb{A}_k^n$ by $((g,v),a) \mapsto ga + v$, called the *standard action*, and it is faithful.

Theorem 1.1. *Let X be an irreducible variety on which $\mathrm{Aff}(V)$ acts faithfully. Assume $\dim(X) \le \dim(V)$. Then X is $\mathrm{Aff}(V)$-isomorphic to V equipped with the standard operation of $\mathrm{Aff}(V)$ on V.*

The conjugacy classes of $\mathrm{Aff}(V)$ acting on itself by conjugation are known, and one can find a description in [Bla06] by JÉRÉMY BLANC. We will use a a similar but more geometric approach to prove that:

Theorem 1.2. *There are finitely many $\mathrm{Aff}(V)$-conjugacy classes in $C_X \times V$. We denote them by $C_{(X,0)}, C_{(X,x_1)}, \ldots, C_{(X,x_s)}$. Then for $i = 1, \ldots, s$*
   (1) $C_{(X,x_i)} = \bigcup_{Y \in C_X} \{Y\} \times (S_Y y_i + Im(Y))$, *where $y_i = gx_i$ for some $g \in \mathrm{GL}(V)$ such that $Y = gXg^{-1}$.*
   (2) $C_X \times V = \overline{C_{(X,x_1)}} \supset \cdots \supset \overline{C_{(X,x_s)}} \supset \overline{C_{(X,0)}} = C_{(X,0)}$.

For any algebraic group G, let $\pi_{G,N} : \mathfrak{g}^{\oplus N} \to \mathfrak{g}^{\oplus N} /\!/ G$ the algebraic quotient of the adjoint action of G on N copies of its Lie algebra $\mathfrak{g}$. Let $\mathrm{SAff}(V) \subset \mathrm{Aff}(V)$ be the subgroup consisting of elements of determinant one.

Theorem 1.3. *Let $pr : \mathfrak{aff}(V)^{\oplus N} \to \mathfrak{gl}(V)^{\oplus N}$. Then*
   (1) $\pi_{\mathrm{Aff}(V),1} = \pi_{\mathrm{GL}(V),1} \circ pr$
   (2) $\pi_{\mathrm{SAff}(V),1} = \pi_{\mathrm{SL}(V),1} \circ pr$
   (3) $\pi_{\mathrm{Aff}(V),N} = \pi_{\mathrm{GL}(V),N} \circ pr$
   (4) *If $\dim(V) = 2$, then $\pi_{\mathrm{SAff}(V),2}$ does not factor through $\pi_{\mathrm{SL}(V),2}$.*

We give the ring of invariants $\mathcal{O}(\mathfrak{saff}_2^{\oplus 2})^{\mathrm{SAff}_2}$ explicitly and thereby show that there exist invariants depending entirely on the translations V^+.

I thank my advisor Prof. Dr. Hanspeter Kraft for his excellent supervision, many useful and elegant ideas, interesting discussions and constant support.

2. PRELIMINARIES AND TERMINOLOGY

We assume the reader to be familiar with basic concepts and results from affine algebraic geometry. For an affine variety X, denote by $\mathcal{O}(X)$ its coordinate ring, and $\mathcal{V}_X(f_1, \ldots, f_n)$ the zero set of $f_1, \ldots, f_n \in \mathcal{O}(X)$.

$\mathrm{Aff}(V)$ is the semi-direct product of the linear maps $\mathrm{GL}(V)$ and the normal subgroup of translations V^+,

$$\mathrm{Aff}(V) = V^+ \rtimes \mathrm{GL}(V) \quad \text{where} (g,v)(a) = ga + v,$$

and the product is defined by
$$(v, g)(w, h) = (gw + v, gh).$$
We will often use the identification $g = (0, g)$ for $g \in \mathrm{GL}(V)$ and $v = (e, v)$ for $v \in V^+$. In case $V = k^n$ we shortly write $\mathrm{Aff}_n = \mathrm{Aff}_n(k) = (k^n)^+ \ltimes \mathrm{GL}_n$. We embed $\mathrm{Aff}(V)$ into $GL(V \oplus k)$ in the usual way: $(v, g) \mapsto \begin{bmatrix} g & v \\ 0 & 1 \end{bmatrix}$.

The Lie algebra of $\mathrm{Aff}(V)$ is given by $\mathfrak{aff}(V) = \mathrm{GL}(V) \oplus V$ which embeds into $\mathfrak{gl}(V \oplus k)$ by $(X, x) \mapsto \begin{bmatrix} X & x \\ 0 & 0 \end{bmatrix}$. It follows that the Lie brackets is

$$[(X, x), (Y, y)] = (XY - YX, Xy - Yx)$$

and the adjoint action of $\mathrm{Aff}(V)$

$$\mathrm{Ad}(v, g)(X, x) = (gXg^{-1}, gx - gXg^{-1}v)$$

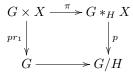In particular, $\mathrm{Ad}(g)(X, x) = (gXg^{-1}, gx)$ and $\mathrm{Ad}(v)(X, x) = (X, x - Xv)$.

The subgroup of $\mathrm{Aff}(V)$ consisting of element of determinant one is denoted by $\mathrm{SAff}(V)$. It is isomorphic to $V \rtimes \mathrm{SL}(V)$, and its Lie algebra is $\mathfrak{saff}(V) = \mathfrak{sl}(V) \oplus V$.

2.1. **Associated bundles.** Let G be an algebraic group, $H \subset G$ a closed subgroup and X an H-variety. On $G \times X$ we consider the following action of H:

$$h \cdot (g, x) := (gh^{-1}, hx)$$

The action is free and all H-orbits are closed and isomorphic to H.

Proposition 2.1. *The quotient $\pi : G \times H \to (G \times X)/\!\!/H$ exists, is a smooth geometric quotient and will be denoted by $G *_H X$. The projection $pr_1 : G \times X \to G$ induces a morphism $p : G *_H X \to G/H$ such that the diagram*
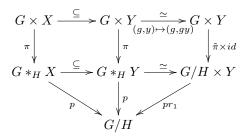
$$
\begin{array}{ccc}
G \times X & \xrightarrow{\ \pi\ } & G *_H X \\
{\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle p} \\
G & \longrightarrow & G/H
\end{array}
$$

is cartesian.

The map $p : G *_H X \to G/H$ is called the *associated bundle* of the principal bundle $G \to G/H$. The name bundle refers to the fact that p is locally trivial in the *étale topology*. If X is a vector space V with a linear action of H one can show that p is a vector bundle, hence locally trivial in the Zariski-topology. We will not give a proof of the proposition here.

Some special cases. Assume that X is an H-stable locally closed subset of a G-variety Y. Consider the following commutative diagram:

$$
\begin{array}{ccccc}
G \times X & \xrightarrow{\subseteq} & G \times Y & \xrightarrow[\ (g,y)\mapsto(g,gy)\ ]{\simeq} & G \times Y \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tilde{\pi}\times id} \\
G *_H X & \xrightarrow{\subseteq} & G *_H Y & \xrightarrow{\simeq} & G/H \times Y \\
& {\scriptstyle p}\searrow & \downarrow{\scriptstyle p} & \swarrow{\scriptstyle pr_1} & \\
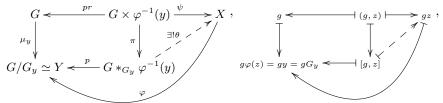& & G/H & &
\end{array}
$$

It follows that $G *_H Y$ is the trivial fibre bundle over G/H with fibre Y and that $G *_H X$ is locally closed in $G *_H Y$. From this one can deduce that $p : G *_H X \to G/H$ is a bundle with fibre X.
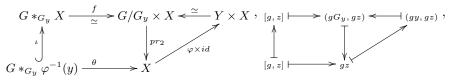
If $H \subset G$ is a reductive subgroup and X an affine H variety, then the quotient $(G \times X)/\!\!/H$ exists as an affine variety with coordinate ring $\mathcal{O}(G \times X)^H$ and is equal to the orbit space $G *_H X$ since all the orbits are closed (see subsection **??**). It follows from the LUNA's Slice Theorem (see f.e. [NE]) that $p : (G \times X)/\!\!/H \to G/H$ is a bundle with fibre X.

Theorem 2.2. *Let G be an algebraic group and let $\varphi : X \to Y$ be a G-equivariant morphism of G-varieties. Assume that $Y = Gy$ for some $y \in Y$. Then there is a canonical G-isomorphism*

$$G *_{G_y} \varphi^{-1}(y) \xrightarrow{\simeq} X, \quad [g, z] \mapsto gz.$$

Proof. The associated bundle is well defined since $\varphi^{-1}(y)$ is a G_y-stable closed subset of the G-variety X. Let $\psi : G \times \varphi^{-1}(y) \to X$, $(g, z) \mapsto gz$, which is a H-invariant morphism. Consider the commutative diagram

$$
\begin{array}{c}
\text{[commutative diagram]}
\end{array}
$$

where $\pi : G \times \varphi^{-1}(y) \to G *_{G_y} \varphi^{-1}(y)$, $(g, z) \mapsto [g, z]$ is the quotient map, p the bundle morphism, pr the canonical projection and μ_y the orbit map. Since π is the quotient map there exists a unique G_y-morphism $\theta : G *_{G_y} \varphi^{-1}(y) \to X$ such that $\theta \circ \pi = \psi$. From the diagram it follows that θ is injective and also surjective since $X = \varphi^{-1}(Gy) = G\varphi^{-1}(y)$ implies that ψ is surjective. It now is left to show that θ^{-1} is a morphism: From the commutative diagram

$$
\begin{array}{c}
\text{[commutative diagram]}
\end{array}
$$

it follows that it is enough to show that $pr_2 \colon f(\iota(G *_{G_y} \varphi^{-1}(y))) \longrightarrow X$ is an isomorphism. In fact, its inverse is given by $x \mapsto (\varphi(x), x)$ which is obviously a morphism. $\qquad\square$

A geometric meaning of this theorem can be seen in the following way:

Corollary 2.3. *There is a bijection*

$$\{G\text{-orbit on } X\} \xleftrightarrow{bij} \{G_y\text{-orbit on } \varphi^{-1}(y)\},$$
$$Gx \longmapsto Gx \cap \varphi^{-1}(y)$$
$$Gz \longleftarrow\!\!\shortmid G_y z$$

Moreover, the bijection respects the orbit closure.

2.2. **Complete varieties.** To define completeness, a short introduction of a more general concept of a variety is necessary and we will follow the definitions of [Mum99]. We remark that [Mum99] only covers irreducible varieties. For non-irreducible varieties see for example [Hum75].

A variety is a prevariety which satisfies the Hausdorff axiom, i.e. $\Delta(X) = \{(x, x) : x \in X\}$ is closed in $X \times X$ with respect to the Zariski-topology on $X \times X$. A prevariety is a connected topological space X together with a sheaf $\mathcal{O}_X$ of k-valued functions on X and with a finite open covering $\{U_i\}$ of X such that U_i together with the restricted sheaf $(\mathcal{O}_X)_{U_i}$ is an affine variety.

A continuous map $\varphi : X \to Y$ is a morphism if and only if for every open subset $U \subset Y$ and $f \in \mathcal{O}_Y(U)$, $f \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(U))$.

Affine and projective varieties are varieties, and for affine varieties the above definition of morphisms coincides with the definition of morphisms between affine varieties [Mum99].

Definition 2.4. A variety X is *complete* if for all varieties Y, the projection morphism $pr_2 : X \times Y \to Y$ is a closed map.

A complete variety has the following properties (cf.[Mum99]):

(i) Let $f : X \longrightarrow Y$ be a morphism where X is complete. Then $f(X)$ is closed in Y and complete.
(ii) An affine variety is complete only if its dimension is zero, i.e. it is a finite sets of points.

Theorem 2.5 (cf.[Mum99])**.** $\mathbb{P}^n$ *is complete for every n.*

Corollary 2.6. *For $n \geq 1$, there is no injective morphism from $\mathbb{P}^n$ into an affine variety.*

3. FAITHFUL $\mathrm{Aff}(V)$-OPERATION ON IRREDUCIBLE AFFINE VARIETIES OF DIMENSION n

Theorem 3.1. *Let $\dim V = n$ and let X be an irreducible variety on which $\mathrm{Aff}(V)$ acts faithfully. Then $\dim X \geq n$.*
If $\dim X = n$, then X is $\mathrm{Aff}(V)$-isomorphic to V equipped with the standard operation of $\mathrm{Aff}(V)$ on V.

The first part of the theorem follows from the fact that if $\mathrm{Aff}(V)$ acts faithfully on X, then also T_n acts faithfully on X together with the following lemma:

Lemma 3.2 (see [Kraft11]). *Let X be an irreducible affine variety on which T_n acts faithfully. Then the set of points with trivial stabiliser is open and dense in X. In particular, X has dimension at least n.*

Proof. First, let X be a vector space V of dimension d and $\rho : T_n \longrightarrow \mathrm{GL}(V)$ a faithful representation of T_n. There exist characters $\chi_1, \ldots, \chi_d$ of T_n and a suitable basis of V such that for every $t \in T_n$, $\rho(t)$ is the diagonal matrix $\mathrm{diag}(\chi_1(t), \ldots, \chi_d(t))$.

Let $\mathfrak{A} := \{v \in V : \mathrm{Stab}_{T_n}(v) \neq \{e\}\}$. Then $\mathfrak{A} \subset \bigcup_{i=1}^{n} k^i \times \{0\} \times k^{n-1-i}$. Let $I \subset \{0,1\}^n$ and $H_\alpha = \prod_{i=1}^{n} B_{\alpha_i}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $B_0 = \{0\}$ and $B_1 = k$. For any $v \in \mathfrak{A}$ and any diagonal matrix D, we have $Dv \in \mathfrak{A}$. Hence $\mathfrak{A} = \bigcup_{\alpha \in I} H_\alpha$. Thus $\mathfrak{A}$ is closed.
Observe that H_α is T-stable and the action of T on H_α is not faithful for any α.

Now, let X be a irreducible affine variety on which T_n acts faithfully. Since T_n acts faithfully on X there exists an injective T_n-equivariant morphism $\theta : X \longrightarrow \mathbb{C}^n$ such that $\theta(X)$ is closed and T_n-stable and X is T_n-isomorphic to $\theta(X)$. Assume that $\theta(X) \subset \mathfrak{A}$. Since X is irreducible there exists $\alpha \in I$ such that $\theta(X) \subset H_\alpha$. The action of T_n on $\theta(X)$ is faithful and hence the action of T_n on H_α is faithful as well, which is a contradiction to the above observation. It follows that $\mathfrak{A} \cap \theta(X) \subsetneq \theta(X)$ is a closed subset and hence the set of points of X with trivial stabilizer is open and dense dense in X. $\qquad \square$

The proof needs the two following lemma Lemma 3.3. Consider the split exact sequence

$$0 \longrightarrow \mathrm{Aff}(V) \overset{p}{\longrightarrow} \mathrm{GL}(V) \longrightarrow 0$$

where we fix a section, i.e. an embedding $GL(V) \hookrightarrow \mathrm{Aff}(V)$.

Lemma 3.3. *Let $G \subset \mathrm{Aff}(V)$ be a closed subgroup. Then the canonical morphism $\bar{p} : \mathrm{Aff}(V)/G \to \mathrm{GL}(V)/p(G)$ has the structure of a $\mathrm{GL}(V)$-vector bundle.*

Proof. Let $G' := p(G) \subset GL(V)$. The morphism $\bar{p} : \mathrm{Aff}(V)/G \to GL(V)/G'$ is $GL(V)$-equivariant. The subgroup $W := V^+ \cap G$ is a subvector space of V which is stable under G'. Moreover, for $(v,g) \in G$ and $w \in W$, we have $W \ni (v,g)w(v,g)^{-1} = gw$. It follows that $\bar{p}^{-1}(eG') = V/W$ has the structure of a vector space with a linear action of G'. Hence, by Theorem 2.2,

$$\mathrm{Aff}(V)/G \simeq \mathrm{GL}(V) *_{G'} V/W,$$

and this is a $GL(V)$ bundle over $GL(V)/G'$ as explained in the subsection "Associated bundles". $\quad \square$

Proof of Theorem 3.1. Let X be an affine variety of dimension $\dim(X) = n := \dim(V)$ with a faithful action of $\mathrm{Aff}(V)$. Then a maximal torus $T \subset \mathrm{GL}(V)$ acts also faithfully, hence has a dense orbit Tx_0 by Lemma 3.2. It follows that the stabiliser $G := \mathrm{Stab}_{\mathrm{Aff}(V)}(x_0)$ has codimension n, i.e. $\dim(\mathrm{Aff}(V)/G) = n$. With the notation of Lemma 3.3 we obtain that

$$(\star) \qquad\qquad n = \dim(V/W) + \dim(\mathrm{GL}(V)/G'),$$

where $W = V^+ \cap G = \mathrm{Stab}_{V^+}(x_0)$. Since V^+ acts faithfully on X, we have $W \subsetneq V^+$.

We claim that $W = (0)$, or, equivalently, that $G' = \mathrm{GL}(V)$. W is G'-stable, hence G' is a subgroup of $\mathrm{Stab}_{\mathrm{GL}(V)}(W)$. If $W \neq (0)$, then G' is contained in the in the parabolic subgroup $P = P_W$ normalising W. Thus $\dim(G') \leq \dim(P) = n^2 - (n-d)d$, where $d = \dim(W) < n$, and so $\dim(\mathrm{GL}(V)/G') \geq d(n-d)$. With $(\star)$, it follows that

$$n = (n-d) + \dim(\mathrm{GL}(V)/G') \geq (d+1)(n-d)$$

and so $n = d + 1$ and hence $G' = P_W$. Therefore, by Lemma 3.3, $\mathrm{Aff}(V)/G$ is a vector bundle over the projective variety $\mathrm{GL}(V)/P_W$, which is a contradiction, because $\mathrm{Aff}(V)/G$ is an orbit and hence quasi-affine.

Since $W = \mathrm{Stab}_{V^+}(x_0) = \{e\}$, the orbit map $V^+ \to X$ is an $\mathrm{Aff}(V)$-equivariant isomorphism, and G is isomorphic to $\mathrm{GL}(V)$. Since $\mathrm{GL}(V)$ is a fully reducible subgroup of $\mathrm{Aff}(V)$, G and $\mathrm{GL}(V)$ are conjugate [Most]. In fact, there exist $v \in V^+$ such that $vGv^{-1} = \mathrm{GL}(V)$. By putting $x'_0 = v^{-1}x_0$, we can assume that $G = \mathrm{GL}(V)$. Lemma 3.3 implies that the canonical morphism $V \to \mathrm{Aff}(V)/GL(V)$ is also an $\mathrm{Aff}(V)$-equivariant isomorphism. This proves the claim. $\qquad\square$

4. The conjugacy classes in $\mathrm{Aff}(V)$; A geometric approach

Consider the adjoint action of $\mathrm{Aff}(V)$ on its Lie algebra $\mathfrak{aff}(V) = \mathfrak{gl}(V) \oplus V$, which is given by $(g, v) \cdot (X, x) := \mathrm{Ad}(g, v)(X, x) = (gXg^{-1}, gx - gXg^{-1}v)$.

4.1. The conjugacy classes. Denote by $C_{(X,x)}$ (resp. C_X) the orbit of $\mathrm{Aff}(V)$ (resp. $\mathrm{GL}(V)$) on $\mathfrak{aff}(V)$ (resp. $\mathfrak{gl}(V)$). Remark that for any $(X, x) \in \mathrm{Aff}(V)$, and any (Y, y) conjugate to (X, x), X and Y are conjugate in $\mathfrak{gl}(V)$ under $\mathrm{GL}(V)$, hence $C_{(X,x)} \subset C_X \times V$ and the latter is covered by the conjugacy classes of the form $C_{(X,x)}$. Observe that the action of $\mathrm{GL}(V)$ on $\mathfrak{gl}(V)$ can be extended to an action of $\mathrm{Aff}(V)$ by defining $(g, v) \cdot X := gXg^{-1}$. It is easy to see that

$$S_X := \mathrm{Stab}_{\mathrm{Aff}(V)}X = \mathrm{Stab}_{\mathrm{GL}(V)}X \times V.$$

The split exact sequence

$$0 \longrightarrow V^+ \longrightarrow \mathrm{Aff}(V) \overset{pr}{\longrightarrow} \mathrm{GL}(V) \longrightarrow 0$$

induces a projection $pr : C_{(X,x)} \longrightarrow C_X$ for every $(X, x) \in \mathrm{Aff}(V)$ which by theorem 2.2 implies that we have an $\mathrm{Aff}(V)$-equivariant isomorphism

$$C_{(X,x)} \simeq \mathrm{Aff}(V) *_{S_X} pr^{-1}(X).$$

Corollary 2.3 applied to our situation gives the following bijection respecting the closure of the orbits:

$(\star)$ $\qquad\qquad \{\mathrm{Aff}(V)\text{-orbit on } C_X \times V\} \overset{\mathrm{bij}}{\longleftrightarrow} \{S_X\text{-orbit on } V\}$

By construction, the action of S_X on V has the following description:

$$(g, v) \cdot x = gx - Xv$$

and so the S_X-orbit is given by $S_X \cdot x + \mathrm{Im}(X)$. Hence there is a bijection

$(\star\star)$ $\qquad\qquad \{S_X\text{-orbits on } V\} \overset{\mathrm{bij}}{\longleftrightarrow} \{S_X\text{-orbits on } V/\mathrm{Im}(X)\}$

which again respects the closures of the orbits. This has the following consequence:

Lemma 4.1.
  (1) $V \times C_X$ is a single conjugacy class if and only if X is invertible. It is closed if and only if X is semisimple.
  (2) $C_{(X,0)} \simeq \mathrm{Aff}(V) *_{S_X} \mathrm{Im}(X)$ is closed in $C_X \times V$, it is a vector bundle over C_X and is contained in all conjugacy classes in $C_X \times V$. This class is closed if and only if X is semisimple.

Proof. We will only prove that fact that $pr : C_{(X,0)} \to C_X$ is a vector bundle over C_X since everything else is a consequence of the above.
Let $\mathrm{rk}(X) := r$ and let $U_{i_1,\ldots,i_r} := \{Y \in C_X : Y_{i_1}, \ldots, Y_{i_r} \text{ are linearly independent}\}$, where Y_{i_k} is the i_kth column of Y. $U_{i_1,\ldots,i_r}$ is a special open set in C_X and the family $\{U_{i_1,\ldots,i_r}\}_{i_1,\ldots,i_r}$ covers C_X. Let $Y \in U_{i_1,\ldots,i_r}$. Then $\mathrm{Im}(Y)$ is spanned by $Y_{i_1}, \ldots, Y_{i_r}$. Define

$$\varphi_{i_1,\ldots,i_r} : pr^{-1}(U_{i_1,\ldots,i_r}) \longrightarrow U_{i_1,\ldots,i_r} \times \mathbb{C}^r, \quad (Y, \sum_{k=1}^{r} a_k Y_{i_k}) \longmapsto (Y, \sum_{k=1}^{r} a_k e_k)$$

where e_k is the kth standard vector in $\mathbb{C}^r$. $\varphi_{i_1,\ldots,i_r}$ is an isomorphism of affine varieties which induces a vector space isomorphism on the fibres. $\qquad\square$

In order to describe the S_X-orbits on $V/\operatorname{Im}(X)$ of a non-invertible element X, and find representatives of them, we make the following construction:

Denote by $V_0 \subset V$ the generalized eigenspace of the eigenvalue 0, i.e. $V_0 := \ker(X^n)$. Then $V = V_0 \oplus V'$ where $V' := \operatorname{Im}(X^n)$. Then $V/\operatorname{Im}(X) \simeq V_0/\operatorname{Im}(X_0)$ where $X_0 := X|_{V_0}$, which is nilpotent. By construction the S_X-orbits on $V/\operatorname{Im}(X)$ coincide with the S_{X_0}-orbits on $V_0/\operatorname{Im}(X_0)$ where $S_{X_0} := \operatorname{Stab}_{GL(V_0)} X_0$. This reduces to the case where X_0 is nilpotent.

Now, let $n_1 > \cdots > n_s \geq 1$ be the different sizes of Jordan blocks of X_0. For every i choose a vector $x_i \in V_0$ such that $x_i \notin \operatorname{Im}(X_0)$, $X_0^{n_i} x_i = 0$ and $X_0^{n_i-1} x_i \neq 0$.

Theorem 4.2. *There are finitely many* $\operatorname{Aff}(V)$*-conjugacy classes in* $C_X \times V$*, denoted by* $C_{(X,0)}, C_{(X,x_1)}, \ldots, C_{(X,x_s)}$*. Then*

(1) $C_X \times V = \overline{C_{(X,x_1)}} \supset \cdots \supset \overline{C_{(X,x_s)}} \supset \overline{C_{(X,0)}} = C_{(X,0)}$.

(2) *For* $i = 1, \ldots, s$

$$C_{(X,x_i)} = \bigcup_{Y \in C_X} \{Y\} \times (S_Y y_i + Im(Y))$$

where $y_i = gx_i$ *for some* $g \in \operatorname{GL}(V)$ *such that* $Y = gXg^{-1}$.

Proof. Assume that X_0 has k_i Jordan blocks of size n_i. Then $\dim V_0 = \sum_i r_i n_i$ and $\dim V_0/\operatorname{Im}(X_0) = \sum_i r_i$. A short calculation shows that the image of S_{X_0} is the subgroup $P \subset GL(V_0/\operatorname{Im}(X_0))$ isomorphic to the subgroup of $GL_{\sum_i r_i}$ consisting of lower triangular matrices which have invertible blocks of size $r_1, \ldots, r_s$ on their diagonal. It is now easy to see that the P-orbits in $V_0/\operatorname{Im}(X_0)$ are given by $x_1, \ldots, x_s$. From this, $(\star)$ and $(\star\star)$ it follows immediately that

$$C_{(X,x_i)} = C_{(X,x_i)} = \bigcup_{Y \in C_X} \{Y\} \times (S_Y y_i + \operatorname{Im}(Y))$$

for every i and

$$C_X \times V = \overline{C_{(X,x_1)}} \supset \cdots \supset \overline{C_{(X,x_s)}} \supset \overline{C_{(X,0)}}.$$

$C_{(X,0)} = C_X \times \operatorname{Im}(X)$ is shown by simply calculating it: For every $Y \in C_X$, (Y, Yx) is conjugate to $(X, 0)$. The other inclusion is trivial. $\qquad\square$

5. Invariants of the adjoint respresentation of $\operatorname{Aff}(V)$ and SAff_n

Denote by $\operatorname{SAff}(V)$ the set of elements of $\operatorname{Aff}(V)$ having determinant one. Its Lie algebra $\mathfrak{saff}(V)$ is isomorphic to $\mathfrak{sl}(V) \oplus V$ and the adjoint action of $\operatorname{SAff}(V)$ is the restriction of the adjoint action of $\operatorname{Aff}(V)$ in the sense that

$$\operatorname{Ad}_{\operatorname{SL}(V)}(g,v)(X,x) = \operatorname{Ad}_{\operatorname{Aff}(V)}(g,v)(X,x)$$

for any $(X,x) \in \mathfrak{saff}(V)$, $(g,v) \in \operatorname{SAff}(V)$.

Let $\pi_G : \mathfrak{g} \to \mathfrak{g}/\!/G$ be the quotient of the adjoint action of G on its Lie algebra $\mathfrak{g}$. In this chapter we will present the quotients $\pi_{\operatorname{Aff}(V)}$ and $\pi_{\operatorname{SAff}(V)}$.

Denote by $pr : \mathfrak{aff}(V) \to \mathfrak{gl}(V)$ the canonical projection onto $\mathfrak{gl}(V)$.

5.1. **The invariants of the adjoint representation of** $\operatorname{Aff}(V)$. Let $(X,x) \in \mathfrak{aff}(V)$. Then $(g,0) \cdot (X,x) = (gXg^{-1}, gx) = (X, gx)$ for any g in the center of $\operatorname{GL}(V)$. Let f be an invariant. Then $f(X,x) = f(X, gx)$ for every g in the center of $\operatorname{GL}(V)$. Thus f is constant on the set $\{X\} \times \mathbb{C}^* x$ and hence also constant on $\{X\} \times \mathbb{C}x$, which means that $f(X,x) = f(X,0)$. Since this holds for any $(X,x) \in \mathfrak{aff}(V)$, it implies that any invariant is an invariant of the adjoint action of $\operatorname{GL}(V)$, i.e. any invariant morphism factors through the quotient $\mathfrak{aff}(V) \to \mathfrak{gl}(V)$.

Theorem 5.1.

$$\mathfrak{aff}(V) \xrightarrow{\ pr\ } \mathfrak{gl}(V) \xrightarrow{\ \pi_{GL(V)}\ } \mathfrak{gl}(V)/\!/\operatorname{GL}(V)$$

is the quotient of the adjoint action of $\operatorname{Aff}(V)$ *on* $\mathfrak{aff}(V)$*, i.e.*

$$\pi_{\operatorname{Aff}(V)} = \pi_{\operatorname{GL}(V)} \circ pr.$$

The proof uses the following well known lemma.

Theorem 5.2 (Richardson Lemma [RL]). *Let G be an algebraic group and $\varphi : X \longrightarrow Y$ a morphism of affine varieties. Assume that Y is normal and*

(1) $codim_Y \overline{Y \backslash \varphi(X)} \geq 2$.

(2) *There is a dense open set $U \subset Y$ such that for every $y \in U$ the fibre $\varphi^{-1}(y)$ contains a dense orbit.*

Then φ is the quotient.

Remark 5.3. In the above theorem the condition (2) can be replaced by the following condition $(2')$: There is an open set $U \in Y$ such that every fibre over U contains a unique closed orbit [PV94].

Proof of theorem 5.1. It suffices to verify (1) and (2) for $\pi_{\mathrm{Aff}(V)}$:
Remember that the quotient of the adjoint action of $\mathrm{GL}(V)$ on $\mathfrak{gl}(V)$ is given by

$$\pi_{\mathrm{GL}(V)} : \mathfrak{gl}(V) \longrightarrow \mathbb{C}^n, \quad X \mapsto (a_0(X), \ldots, a_{n-1}(X))$$

where $\chi_X(t) = t^n - a_{n-1}(X)t^{n-1} + \cdots + (-1)^n a_0(X)$ is the characteristic polynomial of X. Observe that $\chi_{(X,x)}(t) = t\chi_X(t)$ for every $(X,x) \in \mathrm{Aff}(V)$. $\pi_{\mathrm{Aff}(V)}$ is surjective and $\pi_{\mathrm{Aff}(V)}^{-1}(a_0, \ldots, a_{n-1}) = pr^{-1}(\overline{C_X}) = \overline{C_X} \times V$ for some $X \in \mathfrak{gl}(V)$. By theorem 4.2, every fibre contains a dense orbit. $\square$

5.2. The invariants of the adjoint respresentation of $\mathrm{SAff}(V)$.

Theorem 5.4.

$$\mathfrak{saff}(V) \xrightarrow{pr} \mathfrak{sl}(V) \xrightarrow{\pi_{\mathrm{SL}(V)}} \mathfrak{sl}(V) /\!\!/ \mathrm{SL}(V)$$

is the quotient by the adjoint action of $\mathrm{SAff}(V)$ on $\mathfrak{saff}(V)$, i.e.

$$\pi_{\mathrm{SAff}(V)} = \pi_{\mathrm{SL}(V)} \circ pr.$$

Proof. Let $n = \dim V$. For $n = 1$ the proposition is clear. For $n > 1$, we first show that any $\mathrm{SAff}(V)$-invariant morphism $\mathfrak{saff}(V) \to Z$ factors through pr: Let $(X, x) \in \mathfrak{saff}(V)$, which is a subset of $\mathfrak{aff}(V)$. The $\mathrm{SAff}(V)$-orbits on $\mathfrak{saff}(V)$ are classified exactly the same way as the $\mathrm{Aff}(V)$-orbits on $\mathfrak{aff}(V)$ and are given by some $(X, x_1), \ldots, (X, x_s), (X, 0)$ and $\overline{C_{(X,x_1)}} = \overline{C_X} \times V$ just as in the classification of the $\mathrm{Aff}(V)$-orbits in Theorem 4.2 (even though the $\mathrm{SAff}(V)$-orbits and $\mathrm{Aff}(V)$-orbits might not be equal since the orbits of $\mathrm{Stab}_{\mathrm{SL}(V)}X$ and $\mathrm{Stab}_{\mathrm{GL}(V)}X$ on V might not be equal). Hence any $\mathrm{SAff}(V)$-invariant morphism is constant on a dense subset of $\overline{C_X} \times V$, hence constant on $\{X\} \times V$.

Let $\pi_{\mathrm{SAff}(V)} : \mathfrak{saff}(V) \to Q$ be the quotient by $\mathrm{SAff}(V)$. By the above exists an $\mathrm{SAff}(V)$-invariant morphism $\theta : \mathfrak{sl}(V) \to Q$ such that $\pi_{\mathrm{SAff}(V)} = \theta \circ pr$. Consider the following commutative diagram:

$\pi_{\mathrm{SL}(V)} \circ pr$ is $\mathrm{SAff}(V)$-invariant hence there exists a unique $\mathrm{SAff}(V)$-invariant morphism $\sigma : Q \to \mathfrak{sl}(V) /\!\!/ \mathrm{SL}(V)$ such that $\pi_{\mathrm{SL}(V)} \circ pr = \sigma \circ \pi_{\mathrm{SAff}(V)}$. θ is in particular $\mathrm{SL}(V)$-invariant hence there is a unique $\mathrm{SL}(V)$-invariant morphism $\rho : \mathfrak{sl}(V) /\!\!/ \mathrm{SL}(V) \to Q$ such that $\theta = \rho \circ \pi_{\mathrm{SL}(V)}$. It follows that σ and ρ are inverse to each other, making ρ $\mathrm{SAff}(V)$-invariant. This proves the claim. $\square$

6. Invariants of the adjoint action of $\mathrm{Aff}(V)$ and SAff_n on several copies of $\mathrm{Aff}(V)$ and $\mathfrak{saff}_n$ respectively

For any algebraic group G, we can canonically extend the adjoint action on its Lie algebra $\mathfrak{g}$ onto an action on $\mathfrak{g}^{\oplus N}$, by

$$\mathrm{Ad}(g)(Y_1, \ldots, Y_n) = (\mathrm{Ad}(g)(Y_1), \ldots, \mathrm{Ad}(g)(Y_N))$$

for any $g \in G$, $(Y_1, \ldots, Y_N) \in \mathfrak{g}^{\oplus N}$. By abuse of notation, we will call it the adjoint action of G on $\mathfrak{g}^{\oplus N}$. Denote by $\pi_{G,N} : \mathfrak{g}^{\oplus N} \to \mathfrak{g}^{\oplus N} /\!\!/ G$ the quotient of the adjoint action of G on $\mathfrak{g}^{\oplus N}$.

We ask whether the invariants of the adjoint action of $\mathrm{Aff}(V)$ and $\mathrm{SAff}(V)$ on several copies of $\mathrm{Aff}(V)$ and $\mathfrak{saff}(V)$ are independent of the translations as well. In the case of $\mathrm{Aff}(V)$ the answer is yes. The case of $\mathfrak{saff}(V)$ is more complicate and will be presented only for the (first non-trivial) case of $\dim V = 2$ and two copies of $\mathfrak{saff}(V)$.

6.1. **The adjoint representation on N copies of $\mathrm{Aff}(V)$.**

Theorem 6.1. *For any $N \in \mathbb{N}$,*

$$\mathfrak{aff}(V)^{\oplus N} \xrightarrow{\;pr\;} \mathrm{GL}(V)^{\oplus N} \xrightarrow{\pi_{\mathrm{GL}(V),N}} \mathrm{GL}(V)^{\oplus N} /\!\!/ \mathrm{GL}(V)$$
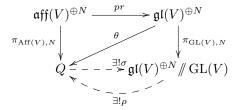
is the quotient by $\mathrm{Aff}(V)$, i.e.

$$\pi_{\mathrm{Aff}(V),N} = \pi_{\mathrm{GL}(V),N} \circ pr.$$

Proof. First, we show that any $\mathrm{Aff}(V)$-invariant morphism $\varphi : \mathfrak{aff}(V)^{\oplus N} \to Z$ factors through pr: Let $(X_1,\ldots,x_N) \in \mathrm{Aff}(V)$. Observe that $(g,0)\cdot(X_1,\ldots,x_N) = (X_1,\ldots,X_N,gx_1,\ldots,gx_N)$ for any g in the center of $\mathrm{GL}(V)$. This means that φ is constant on the set $\{(X_1,\ldots,X_N)\} \times \mathbb{C}^*(x_1,\ldots,x_N)$, hence it is also constant on $\{(X_1,\ldots,X_N)\} \times \mathbb{C}(x_1,\ldots,x_N)$ and so

$$\varphi(X_1,\ldots,X_n,x_1,\ldots,x_N) = \varphi(X_1,\ldots,X_N,0,\ldots,0).$$

Let $\pi_{\mathrm{Aff}(V),N} : \mathfrak{aff}(V)^{\oplus N} \to Q$ be the quotient by $\mathrm{Aff}(V)$. By the above there exists an $\mathrm{Aff}(V)$-invariant morphism $\theta : \mathfrak{gl}(V)^{\oplus N} \to Q$ such that $\pi_{\mathrm{Aff}(V),N} = \theta \circ pr$. $\pi_{\mathrm{GL}(V)} \circ pr$ is $\mathrm{Aff}(V)$-invariant hence there exists a unique $\mathrm{Aff}(V)$-invariant morphism $\sigma : Q \to \mathfrak{gl}(V)/\!\!/\mathrm{GL}(V)$ such that $\sigma \circ \pi_{\mathrm{Aff}(V),N} = \pi_{\mathrm{GL}(V),N} \circ pr$. θ is in particular $\mathrm{GL}(V)$-invariant hence there exists a unique $\mathrm{GL}(V)$-invariant morphism $\rho : \mathfrak{gl}(V)^{\oplus N} /\!\!/ \mathrm{GL}(V) \to Q$ such that $\theta = \pi_{\mathrm{GL}(V),N} \circ \rho$. The situation is summarised in the following diagram:

It follows that σ and ρ are inverse to each other, making ρ $\mathrm{Aff}(V)$-invariant. This proves the claim. $\qquad\square$

6.2. **The adjoint representation on N copies of $\mathfrak{saff}_{\mathrm{n}}$.** Remark that the adjoint action of $\mathrm{SAff}(V)$ induces an action of $(V^N)^+$ and an action of $\mathrm{SL}(V)$ on $\mathfrak{saff}(V)$. Theorem 6.2 states that there are non-trivial V^+-invariants on an open subset of $\mathfrak{saff}(V)^{\oplus 2}$. For $\dim(V) = N = 2$, the quotient of the adjoint action is found by combining the $(V)^+$-invariants and $\mathrm{SL}(V)$-invariants.

Observe that $(\mathfrak{saff}(V)^{\oplus 2})_{\det A \cdot \det B}$ is a V^+-stable subset of $\mathfrak{saff}(V)^{\oplus 2}$.

Theorem 6.2.

$$(\mathfrak{saff}(V)^{\oplus 2})_{\det A \cdot \det B} \longrightarrow (\mathfrak{sl}(V)^{\oplus 2})_{\det A \cdot \det B} \oplus V$$
$$(X,Y,x,y) \longmapsto (X,Y,\ X^{-1}x - Y^{-1}y).$$

is the quotient of $(\mathfrak{saff}(V)^{\oplus 2})_{\det A \cdot \det B}$ by the action of V^+.

Proof. Let

$$\pi : (\mathfrak{saff}(V)^{\oplus 2})_{\det A \cdot \det B} \longrightarrow (\mathfrak{sl}(V)^{\oplus 2})_{\det A \cdot \det B} \oplus V$$
$$(X,Y,x,y) \longmapsto (X,Y,\ X^{-1}x - Y^{-1}y).$$

π is a surjective, $V+$-invariant and $(\mathfrak{sl}(V)^{\oplus 2})_{\det A \cdot \det B} \oplus V$ is normal. The fibres of π have the following form:

$$\pi^{-1}(X,Y,z) = \{(X,Y,x,Y(X^{-1}x - z)) : x \in V\}$$
$$= \{(X,Y)\} \times \mathrm{Graph}(x \mapsto Y(X^{-1}x - z)) \simeq \{(X,Y)\} \times \mathrm{Im}(Y)$$

On the other hand, let $f_{X,Y} : V \longrightarrow V \times V, \ z \mapsto (Xz,Yz)$. Then

$$V^+ \cdot (X,Y,x,y) = \{(X,Y,x - Xz,y - Yz) : z \in V\}$$
$$= \{(X,Y)\} \times ((x,y) - \mathrm{Im}(f)_{X,Y}),$$

$f_{X,Y}$ is linear and injective and so the dimension of the orbit is $n = \dim(V)$. The dimension of the fibre is n as well and since the orbit is contained in the fibre and both are closed and irreducible, they are equal. By theorem 5.2 the claim follows. $\qquad\square$

From now on, let $\dim(V) = 2$.

Lemma 6.3. *Let*

$$\pi^{V^+} : \mathfrak{saff}(V)^{\oplus 2} \to \mathfrak{saff}(V)^{\oplus 2}$$
$$(X, Y, x, y) \mapsto (X, Y, YXx - X^2y, XYy - Y^2x).$$

Then

(1) π *is* V^+*-invariant and* $\mathrm{SL}(V)$*-equivariant.*

(2) $\mathcal{V}(Ab + Ba)_{\mathrm{tr}(AB)} \overset{(a)}{\subset} \mathrm{Im}(\pi^{V^+}) \overset{(b)}{\subset} \mathcal{V}(Ab + Ba)$

Proof. (1): This follows from $X^2 = \det(X)\mathrm{Id}$.
(2): (a) Let $(X, Y, x, y) \in \mathcal{V}(Ab + Ba)_{\mathrm{tr}(AB)}$. We show that

$$\pi(X, Y, \tfrac{x}{\mathrm{tr}(XY)}, \tfrac{y}{\mathrm{tr}(XY)}) = (X, Y, x, y).$$

From $Xy + Yx = 0$ it follows that $-X^2y = XYx$, $YXy = -Y^2x$. Using $\mathrm{tr}(XY)\mathrm{Id} = XY + YX$, this implies that $\mathrm{tr}(XY)x = YXx - X^2y$, $\mathrm{tr}(XY)y = XYy - Y^2x$. Hence

$$x = YX\tfrac{x}{\mathrm{tr}(XY)} - X^2\tfrac{y}{\mathrm{tr}(XY)}, \quad y = XY\tfrac{y}{\mathrm{tr}(XY)} - Y^2\tfrac{x}{\mathrm{tr}(XY)}.$$

(b) $X(XYy - Y^2x) + Y(YXx - X^2y) = X^2Yy - XY^2x + Y^2Xx - YX^2y = 0$. $\qquad\square$

Lemma 6.4. *For any* $X, Y \in \mathfrak{sl}(V)$*, let*

$$\rho_{X,Y} : V \times V \longrightarrow V \times V, \quad (x, y) \mapsto (YXx - X^2y, XYy - Y^2x).$$

(1) *The following subsets of* $\mathfrak{sl}(V)$ *are equal:*

$$\mathcal{V}(A^2, B^2, AB) = \{(X, Y) \in \mathfrak{sl}(V)^{\oplus 2} : \rho_{X,Y} < 2\}$$
$$= \{(X, Y) : \mathrm{Im}\, \rho_{X,Y} \subsetneq \ker((\tilde{x}, \tilde{y}) \mapsto X\tilde{y} + Y\tilde{x})\}$$
$$= \mathcal{V}(\mathrm{tr}(AB), \mathrm{tr}(B^2), \mathrm{tr}(A^2))$$

(2) $\mathfrak{saff}(V)^{\oplus 2} \supset \mathcal{V}(Ab + Ba)$ *has dimension 8.*

(3) $\mathfrak{saff}(V)^{\oplus 2} \supset \mathcal{V}(\mathrm{tr}(AB), \mathrm{tr}(A^2), \mathrm{tr}(B^2), Ab + Ba)$ *has dimension at most six.*

Proof. We will only proof (2) and (3) because (1) is an easy but somewhat long exercise.
(2): $\mathcal{V}(Ab + Ba) = \mathcal{V}(f_1, f_2)$ where $f_1 = a_{11}b_1 + a_{12}b_2 + b_{11}a_1 + b_{12}a_2$ and $f_2 = a_{21}b_1 - a_{11}b_2 + b_{21}a_1 - b_{11}a_2$.
By the Serre-criterion we have $I(\mathcal{V}(Ab + Ba)) = (f_1, f_2)$. By Krull, $\mathcal{V}(f_1)$ has dimension 9. Moreover, it is irreducible since f_1 is irreducible. Thus its coordinate ring $\mathcal{O}(\mathfrak{saff}(V)^{\oplus 2})/(f_1)$ has no zero-divisors and one can easily check that the class of f_2 is non-zero and also non-invertible in $\mathcal{O}(\mathfrak{saff}(V)^{\oplus 2})/(f_1)$. So by Krull, $\dim \mathcal{V}(Ab + Ba) = 8$. (3): Let $\mathcal{V}' := \mathcal{V}(\mathrm{tr}(A^2), \mathrm{tr}(B^2))$. Observe that by the Serre-criterion we have that

$$I(\mathcal{V}') = (\mathrm{tr}(A^2), \mathrm{tr}(B^2)), \; I(\mathcal{V}'(\mathrm{tr}(AB))) = (\mathrm{tr}(A^2), \mathrm{tr}(B^2), \mathrm{tr}(AB)),$$

(+) $\qquad I(\mathcal{V}'(f_1)) = (\mathrm{tr}(A^2), \mathrm{tr}(B^2), f_1).$

Thus $\mathrm{tr}(AB) \notin (\mathrm{tr}(A^2), \mathrm{tr}(B^2), f_1)$. Also it is not invertible in $\mathcal{O}(\mathcal{V}'(f_1))$ and so by Krull it suffices to show that $\mathcal{V}'(f_1)$ is irreducible and has dimension 7.

It is clear that $\mathcal{V}'$ is irreducible and of dimension 8. By (+) we have that $f_1 \notin (\mathrm{tr}(A^2), \mathrm{tr}(B^2))$ and it is also non-invertible in $\mathcal{O}(\mathcal{V}')$ from which it follows that

$$\dim \mathcal{V}(\mathrm{tr}(A^2), \mathrm{tr}(B^2), f_1) = 7.$$

Lets show that the class $\bar{f}_1$ of f_1 in $\mathcal{O}(\mathcal{V}')$ is irreducible: Lets assume that $\mathcal{O}(\mathcal{V}') \ni \bar{f}_1 = gh$ where $g, h \in \mathcal{O}(\mathcal{V}')$. $\mathcal{O}(\mathcal{V}')$ is isomorphic to $\mathcal{O}(\mathfrak{sl}(V))/(\mathrm{tr}(A^2), \mathrm{tr}(B^2))$ and it is an integral domain hence $\deg_{\overline{x_i}} h, \deg_{\overline{x_i}} g \leq \deg_{\overline{x_i}} \bar{f}_1$ for every
$x_i \in \{a_{11}, a_{12}, a_{21}, b_{11}, b_{12}, b_{21}, a_1, a_2, b_1, b_2\}$. It follows that either $\deg_{\overline{a_1}} h = 0$ or $\deg_{\overline{a_1}} g = 0$. Assume

the first. Then $\bar{f}_1 = hg = h(g_0 + g_1\overline{a_1})$ where $\deg_{\overline{a_1}} g_0 = \deg_{\overline{a_1}} g_1 = 0$. Moreover, $\deg_{\overline{b_1}} h = 0$ or $\deg_{\overline{b_1}} g = 0$. The second implies that $\deg_{\overline{b_1}} g_0 = \deg_{\overline{b_1}} g_1 = 0$ and $h = h_0 + h_1 b_1$ where $\deg_{\overline{b_1}} h_0 = \deg_{\overline{b_1}} h_1 = 0$ which implies that $h_1 g_1 = 0$. This means that either $\deg_{\overline{a_1}} h = \deg_{\overline{a_1}} g = 0$ or $\deg_{\overline{b_1}} g = \deg_{\overline{b_1}} h = 0$ which is not possible. Hence $\deg_{\overline{b_1}} h = 0$ and so $f = hg = h(g_{00} + g_{01}\overline{b_1} + g_{10}\overline{a_1})$ for some g_{00}, g_{10}, g_{01} of degree 0 in $\overline{b_1}$. Proceeding like this for $\overline{b_2}$ and $\overline{a_2}$ we get that in fact $\deg_{\overline{b_i}} h = \deg_{\overline{a_i}} h = 0$ for $i = 1, 2$. Thus $\overline{f_1}$ is irreducible and hence $\mathcal{V}'(f_1)$ is irreducible. $\qquad\square$

Lemma 6.5. *Let* $\dim(V) = 2$. *Then*
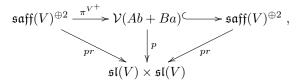
$$\pi^{V^+} : \mathfrak{saff}(V)^{\oplus 2} \to \mathcal{V}(Ab + Ba) \subset \mathfrak{saff}(V)^{\oplus 2}$$

$$(X, Y, x, y) \mapsto (X, Y, YXx - X^2y, XYy - Y^2x)$$

is the quotient by the adjoint action of $\mathrm{SAff}(V)$ *restricted to* V^+.

Proof of Lemma 6.5. Let

$$\pi^{V^+} : \mathfrak{saff}(V)^{\oplus 2} \longrightarrow \mathfrak{saff}(V)^{\oplus 2}$$

$$(X, Y, x, y) \longmapsto (X, Y, YXx - X^2y, XYx - Y^2y)$$

Lemma 6.3 shows that $\operatorname{Im}\pi^{V^+} \subset \mathcal{V}(Ab + Ba)$ and π^{V^+} is $(V)^+$-invariant. Consider the commutative diagramm

$$
\begin{array}{ccccc}
\mathfrak{saff}(V)^{\oplus 2} & \xrightarrow{\ \pi^{V^+}\ } & \mathcal{V}(Ab + Ba) & \hookrightarrow & \mathfrak{saff}(V)^{\oplus 2} \\
& \searrow{\scriptstyle pr} & \downarrow{\scriptstyle p} & \swarrow{\scriptstyle pr} & \\
& & \mathfrak{sl}(V) \times \mathfrak{sl}(V) & &
\end{array}
\;,
$$

where pr is the canonical projection and p its restriction to $\mathcal{V}(Ab + Ba)$. $\mathcal{V}(Ab + Ba)$ is normal by the Serre-criterion. Hence, to prove the claim, it suffices to verify the conditions (1) and (2$'$) in theorem 5.2.

Since $\operatorname{codim}_{\mathcal{V}(Ab+Ba)} \overline{\mathcal{V}(Ab + Ba) \backslash \operatorname{Im}\pi^{V^+}} \geq 2$ is equivalent to $\dim \overline{\mathcal{V}(Ab + Ba) \backslash \operatorname{Im}\pi^{V^+}} \leq 6$ by 6.4 (2), it suffices to show that $\overline{\mathcal{V}(Ab + Ba) \backslash \operatorname{Im}(\pi^{V^+})}$ is contained in a variety of dimensionat most 6. Let $\mathcal{N} := \mathcal{V}(A^2, B^2, AB)$. Observe that by lemma 6.4 (1),

$$p^{-1}(\mathcal{N}) = \mathcal{V}(\operatorname{tr}(A^2), \operatorname{tr}(B^2), \operatorname{tr}(AB), Ab + Ba)$$

which has dimension at most 6 by 6.4 (3). We show that $\overline{\mathcal{V}(Ab + Ba) \backslash \operatorname{Im}\pi^{V^+}} \subseteq p^{-1}(\mathcal{N})$.

$$(X, Y) \notin \mathcal{N} \overset{6.4(1)}{\Leftrightarrow} \operatorname{Im}\rho_{X,Y} = \ker\left((\tilde{x}, \tilde{y}) \mapsto (X\tilde{y} + Y\tilde{x})\right) = \mathcal{V}(X\tilde{y} + Y\tilde{x})$$

$$\Leftrightarrow (x, y) \in \mathcal{V}(X\tilde{y} + Y\tilde{x}) \text{ implies } (X, Y, x, y) \in \operatorname{Im}\rho_{X,Y}$$

$$\Leftrightarrow (X, Y, \alpha, \beta) \in \mathcal{V}(Ab + Ba) \text{ implies } (X, Y, \alpha, \beta) \in \operatorname{Im}\rho_{X,Y}$$

Hence

$$p^{-1}(\mathfrak{sl}(V) \times \mathfrak{sl}(V) \backslash \mathcal{N}) = \{(X, Y, x, y) : (X, Y) \notin \mathcal{N}\}$$

$$\subseteq \{(X, Y, x, y) : (X, Y, x, y) \in \operatorname{Im}\rho_{X,Y}\} = \operatorname{Im}\pi^{V^+}.$$

It follows that $\mathcal{V}(Ab + Ba) \backslash \operatorname{Im}\pi^{V^+} \subseteq \mathcal{V}(Ab + Ba) \backslash p^{-1}(\mathfrak{sl}(V) \times \mathfrak{sl}(V) \backslash \mathcal{N}) = p^{-1}(\mathcal{N})$ and hence $\overline{\mathcal{V}(Ab + Ba) \backslash \operatorname{Im}\pi^{V^+}} \subseteq p^{-1}(\mathcal{N})$.

(2') follows from a short calculation that shows that every fibre of the open set $\mathcal{V}(Ab + Ba)_{\det(A)}$ consists of exactly one orbit, which is closed because $(V)^+$ is unipotent.

$\qquad\square$

Lemma 6.6. *Let* $\dim(V) = 2$. *Let* $\pi^{\mathrm{SL}(V)} : \mathfrak{saff}(V)^{\oplus 2} \to \mathfrak{saff}(V)^{\oplus 2} /\!\!/ \mathrm{SL}(V)$.

$$\mathfrak{saff}(V)^{\oplus 2} \xrightarrow{\ \pi^{V^+}\ } \mathcal{V}(Ab + Ba) \xrightarrow{\ \pi^{\mathrm{SL}(V)}\ } (\mathfrak{saff}(V)^{\oplus 2} /\!\!/ V^+) /\!\!/ \mathrm{SL}(V)$$

is the quotient of the adjoint action of $\mathrm{SAff}(V)$ *on* $\mathfrak{saff}(V)^{\oplus 2}$.

Proof. $\mathfrak{saff}(V)/\!\!/V^+ = \mathcal{V}(Ab+Ba) \subset \mathfrak{saff}(V)^{\oplus 2}$ by Lemma 6.5. $\mathcal{V}(Ab+Ba)$ is $\mathrm{SL}(V)$-stable. It follows that

$$\mathrm{Im}(\pi^{\mathrm{SL}(V)} \circ \pi^{V^+}) = \pi^{\mathrm{SL}(V)}(\mathcal{V}(Ab+Ba)) = \mathcal{V}(Ab+Ba)/\!\!/\mathrm{SL}(V).$$

Let $\pi_{\mathrm{SAff}(V)} : \mathfrak{saff}(V)^{\oplus 2} \to Z$ be the quotient of the adjoint action of $\mathrm{SAff}(V)$. We show that $\mathcal{V}(Ab+Ba)/\!\!/\mathrm{SL}(V) = Z$:

Since $\pi_{\mathrm{SAff}(V)}$ is $\mathrm{SAff}(V)$-invariant, it is also $(V)^+$-invariant. Hence there exists a unique $(V)^+$-invariant morphism $\varphi : \mathcal{V}(Ab+Ba) \to Z$ such that $\varphi \circ \pi^{V^+} = \pi_{\mathrm{SAff}(V)}$. Since $\pi^{\mathrm{SL}(V)} \circ \pi$ is $\mathrm{SAff}(V)$-invariant there exists a unique $\mathrm{SAff}(V)$-invariant morphism $\theta : Z \to \mathcal{V}(Ab+Ba)/\!\!/\mathrm{SL}(V)$ such that $\theta \circ q = \pi^{\mathrm{SL}(V)} \circ \pi^{V^+}$. φ is $\mathrm{SL}(V)$-invariant because $\pi_{\mathrm{SAff}(V)}$ is $\mathrm{SAff}(V)$-invariant and π^{V^+} is $\mathrm{SL}(V)$-equivariant. Since $\pi^{\mathrm{SL}(V)} : \mathcal{V}(Ab+Ba) \to \mathcal{V}(Ab+Ba)/\!\!/\mathrm{SL}(V)$ is the quotient map of the action of $\mathrm{SL}(V)$, there exists a unique $\mathrm{SL}(V)$-invariant morphism $\psi : \mathcal{V}(Ab+Ba)/\!\!/\mathrm{SL}(V) \to Z$ such that $\psi \circ \pi^{\mathrm{SL}(V)} = \varphi$. The situation is summarised in the following commutative diagram:

ψ and θ are inverse to each other, hence the claim follows. $\qquad\square$

The following lemma is very well known. For a reference, see for example [Kraft11].

Lemma 6.7.
$$\mathcal{O}(Q_3^{\oplus 2})^{SO_3} = \mathcal{O}(\mathfrak{sl}_2^{\oplus 2})^{SL_2} = k[tr(A^2), tr(B^2), tr(AB)],$$
where Q_3 is the set of the quadratic froms in three variables.

Remark 6.8. We can see easily that these invariants are also $\mathrm{SAff}(V)$-invariants. Let $a^* = \begin{bmatrix} -a_2 & a_1 \end{bmatrix}$. Then a^*b is $\mathrm{SL}(V)$-invariant.

Theorem 6.9. *Let $\dim(V) = 2$. The ring of invariants $\mathcal{O}(\mathfrak{saff}(V)^{\oplus 2})^{\mathrm{SL}(V)}$ is spanned minimally by the following invariants:*

$$
\begin{array}{llllll}
\text{of degree 2:} & tr(A^2) & tr(AB) & tr(B^2) & a^*b \\
\text{of degree 3:} & a^*Aa & a^*Ab & b^*Ab & a^*Ba & a^*Bb & b^*Bb \\
\text{of degree 4:} & a^*ABa & a^*ABb & b^*ABb
\end{array}
$$

where $a^* = \begin{bmatrix} -a_2 & a_1 \end{bmatrix}$.

To give an idea of the proof, the following remark will provide a nice foundation:

Remark 6.10. $\mathcal{O}(\mathfrak{saff}_2^{\oplus 2}) =: \mathcal{O}$ is normal, hence $\mathcal{O}^{SL_2}$ is normal. Since SL_2 is reductive, $\mathcal{O}^{SL_2}$ is finitely generated. Hence $\mathcal{O}^{SL_2} = \sum_{i=1}^k r_i k[t_1, \dots, t_d]$ for some $r_1, \dots, r_k \in \mathcal{O}^{SL_2}$ and some algebraically independent elements $t_1, \dots, t_d \in \mathcal{O}^{SL_2}$. We can assume $r_1 = 1$. Since $\mathcal{O}^{SL_2}$ has a natural grading (with weights on the generators), $r_2, \dots, r_k$ can be chosen to be homogeneous. Furthermore, the Hochster-Roberts theorem shows that $\mathcal{O}^{SL_2}$ is a Cohen-Macaulay algebra, because SL_2 is linearly reductive and $\mathfrak{saff}_2^{\oplus 2}$ smooth. This implies that the Hilbert series of $\mathcal{O}^{SL_2}$ is of the form

$$\mathrm{HS}(\mathcal{O}^{SL_2}, z) = \frac{z^{p_1} + \cdots + z^{p_k}}{(1 - z^{e_1})(1 - z^{e_d})}$$

where $p_i := \deg t_i$ and $e_i := \deg r_i$. $\mathcal{O}$ is factorial, SL_2 connected and has no nontrivial characters. Hence $\mathcal{O}^{SL_2}$ is factorial as well. A factorial Cohen-Macaulay algebra is aways Gorenstein. $\mathcal{O}^{SL_2}$ being a Cohen-Macaulay algebra and Gorenstein is equivalent to the Hilbert polynomial $h(z) := z^{p_1} + \cdots + z^{p_k}$ being reciprocal, i.e. $h(z) = z^{p_k} h(z^{-1})$. A reciprocal polynomial $p(t) = a_0 + a_1 z + \cdots a_n z^n$ has the nice property that $a_i = a_{n-i}$. Also, $h(z)$ has the property $a_1 \le a_1 \le \cdots \le a_k$, where $k = \frac{n}{2}$ if n is even and $k = \frac{n-1}{2}$ if k is odd.
The theorems used can be found in [PV94].

Idea of the proof of Theorem 6.9. Note that indeed all the thirteen polynomials are SL_2-invariants. Let us call them $f_1, \ldots, f_{13}$. We use the tools of classical invariant theory (a summary of some classical results can be found in e.g. [PV94]) and the aid of mathematica, Lie and singular to do the computations.

Consider the graded algebra $k[z_1, \ldots, z_7]$ where $\deg z_i = 2$ for $i = 1, \ldots, 4$ and $\deg z_i = 3$ for $i = 5, 6, 7$. There is an algebra homomorphism

$$\mathcal{A} := k[z_1, \ldots, z_7] \longrightarrow \mathcal{O}^{SL_2}, \quad z_i \mapsto f_i$$

The Hilbert series of $\mathcal{A}$ is

$$\mathrm{HS}(\mathcal{A}, t) = \frac{1}{(1 - t^2)^4 (1 - t^3)^3}.$$

The action of SL_2 on the summands $\mathcal{O}_j$ of the graded ring $\mathcal{O}$ is well known (cf. [Hil93]) and we can compute $d_j := \dim \mathcal{O}_j^{SL_2}$ for each j. Hence we get the following table

i	0	1	2	3	4	5	6	7	...
$\dim \mathcal{A}_i$	1	0	4	3	10	12	26	30	...
d_i	1	0	4	6	13	24	47	70	...

We can obtain the Hilbert series of $\mathcal{O}^{SL_2}$ from the Hilbert series of $\mathcal{A}$ by adding graded summands to $\mathcal{A}$: Let $z_8, \ldots, z_{13}$ be some new variables with $\deg z_i = 3$ for $i = 8, 9, 10$, $\deg z_i = 4$ for $i = 11, 12, 13$. The table shows that the coefficients of the first seven Hilbert series coefficients of $\mathcal{A} + z_8\mathcal{A} + \cdots z_{13}\mathcal{A}$ and $\mathcal{O}^{SL_2}$ are equal. In fact,

$$\mathrm{HS}(\mathcal{A} + \sum_i z_i\mathcal{A}, t) = \frac{1 + 3t^3 + 3t^4 + t^7}{(1 - t^2)^4 (1 - t^3)^3},$$

and its Hilbert polynomial is reciprocal. Adding any more new variables would result in the Hilbert polynomial of the new algebra not being reciprocal or its coefficients not having the property mentioned in remark 6.10. Hence $\mathcal{O}^{SL_2}$ and $\mathcal{A} + \sum_{i=8}^{13} z_i\mathcal{A}$ have the same Hilbert series and thus the algebra homomorphism

$$\mathcal{A} + \sum_{i=8}^{13} z_i\mathcal{A} \longrightarrow \mathcal{O}^{SL_2}, \quad z_i \mapsto f_i$$

is surjective. Therefore, $\mathcal{O}^{SL_2}$ is generated by $f_1, \ldots, f_{13}$. Moreover, it is minimally generated by them since its Hilbert polynomial is irreducible. $\square$

Corollary 6.11. *Let* $\dim(V) = 2$. *The invariant ring* $\mathcal{O}(\mathfrak{saff}(V)^{\oplus 2})^{\mathrm{SAff}(V)}$ *is spanned by the following invariants modulo the ideal* $(Ab + Ba)^{\mathrm{SL}(V)}$:

$$
\begin{array}{llllll}
\text{of degree 2:} & tr(A^2) & tr(AB) & tr(B^2) & a^*b \\
\text{of degree 3:} & a^*Aa & a^*Ab & b^*Ab & a^*Ba & a^*Bb & b^*Bb \\
\text{of degree 4:} & a^*ABa & a^*ABb & b^*ABb
\end{array}
$$

where $a^* = \begin{bmatrix} -a_2 & a_1 \end{bmatrix}$.

Proof. $(Ab + Ba)$ is a $\mathrm{SL}(V)$-stable ideal and $I(\mathcal{V}(Ab + Ba)) = (Ab + Ba)$. By Lemma 6.6, it follows that

$$\mathcal{O}(\mathfrak{saff}_2)^{\mathrm{SAff}_2} \simeq \mathcal{O}(\mathfrak{saff}(V)^{\oplus 2})^{\mathrm{SL}(V)}/(Ab + Ba)^{\mathrm{SL}(V)}.$$

$\square$

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